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Moyal dynamics and trajectories

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Abstract

We give first an approximation of the operator $\delta_h : f \to \delta_h f := h *_\hbar f - f *_\hbar h$ in terms of $\hbar^{2n}, n \geqslant 0$, where $h \equiv h(p,q), (p,q) \in \mathbb{R}^{2n}$, is a Hamilton function and $*_\hbar$ denotes the star product. The operator, which is the generator of time translations in a $*_\hbar$-algebra, can be considered as a canonical extension of the Liouville operator $L_h : f \to L_h f := \{h, f\}_{\text{Poisson}}$. Using this operator we investigate the dynamics and trajectories of some examples with a scheme that extends the Hamilton–Jacobi method for classical dynamics to Moyal dynamics. The examples we have chosen are Hamiltonians with a one-dimensional quartic potential and two-dimensional radially symmetric nonrelativistic and relativistic Coulomb potentials, and the Hamiltonian for a Schwarzschild metric. We further state a conjecture concerning an extension of the Bohr–Sommerfeld formula for the calculation of the exact eigenvalues for systems with classically periodic trajectories.

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1. Introduction

Quantum mechanics in phase space has been considered under many different aspects. This is very lucidly presented in the excellent monograph by Zachos *et al* [1] which also serves as a very helpful basic reference in general. (For an extensive and continuously updated database of references on this subject see also [2].) We shall here present an approach which is a straightforward extension of classical phase space dynamics via an $\hbar$-deformation in the manner of Moyal together with an extension of the classical Hamilton–Jacobi method. Following an established convention [3] we call this scheme *Moyal dynamics* and the related trajectories (depending on parameters like time and angles) *Moyal trajectories*. For $\hbar \to 0$ these trajectories go over in their classical correspondents. We shall confirm by the way a remark made in [3] which states that Moyal trajectories and its corresponding classical trajectories do in general *not* coincide, thus contradicting a widespread erroneous assumption which claims the contrary to be true. Our approach is not a 'quantization' which uses a Hilbert space [4, 5]. We shall, however, on the basis of our approach, present in section 6 a generalization of the Bohr–Sommerfeld approximation of the eigenvalues of stationary states which we believe to deliver exact energy eigenvalues in a Schrödinger picture. In section 7 we shortly connect our approach with conventional quantum mechanics via Wigner transforms (adding nothing genuinely new).

Now, our approach is as follows. Let $h(p, q)$ be a time-independent Hamilton function of a finite degree of freedom. We define the star product via Fourier transformation [4, 5] by $a(p,q) *_\hbar b(p,q) := a(p+\mathrm{i}\hbar\partial_q/2, q-\mathrm{i}\hbar\partial_p/2)b(p,q)$, where $(p+\mathrm{i}\hbar\partial_q/2, q-\mathrm{i}\hbar\partial_p/2) \equiv (p_1+\mathrm{i}\hbar\partial_{q1}/2, p_2+\mathrm{i}\hbar\partial_{q2}/2, \ldots, q_1-\mathrm{i}\hbar\partial_{p1}/2, q_2-\mathrm{i}\hbar\partial_{p2}/2, \ldots)$. The map defined by

$$f(p,q) \to \delta_h f(p,q) := (\mathrm{i}/\hbar)[h(p,q) *_\hbar f(p,q) - f(p,q) *_\hbar h(p,q)] \tag{1}$$

shall be called a δ_h-*Moyal operator*. In mathematical lingo δ_h is an inner derivation on a noncommutative algebra of functions of (p, q) with a star product. That is, $\delta_h(a *_\hbar b) = (\delta_h a) *_\hbar b + a *_\hbar (\delta_h b)$. Due to this property the maps $\exp(t\delta_h), t \in \mathbb{R}$, form a group of automorphisms on a Moyal algebra of functions of (p, q). This group represents the time evolution of the dynamical system determined by the Hamilton function $h(p, q)$. That is, the time evolution of a function $f(p, q)$ is $f(p,q) \to f(p,q,t) \equiv \exp(t\delta_h)(f(p,q))$. This leads to the equation

$$(\partial_t - \delta_h) f(p,q,t) = 0, \tag{2}$$

which will be the starting point of our program. We note first that δ_h splits up into a sum $L_h + \hbar^2\Delta_h$, $\Delta_h = \Delta_{h,0} + \hbar^2\Delta_{h,1} + \cdots$, where L_h is the Liouville operator. This operator (which is an *outer* derivation on an algebra of functions of (p, q) with the ordinary commutative product) generates the classical time evolution of the system determined by $h(p, q)$. Thus, (2) reads for $\hbar = 0$

$$(\partial_t - L_h) f(p,q,t) = 0. \tag{3}$$

To prepare the investigation of relation (2), we shall first consider (3). As an example we take the Hamilton function $h \equiv \left(p_r^2 + p_\phi^2/r^2\right)/2m_0 + a/r$. The corresponding Liouville operator is

$$L_h = p_\phi\partial_\phi/m_0r^2 + p_r\partial_r/m_0 + \left(p_\phi^2 - ar\right)\partial_{p_r}/r^3. \tag{4}$$

Substituting

$$r = u, \qquad p_r = \sqrt{-p_\phi^2/u^2 + 2am_0/u + 2m_0E}, \qquad E \equiv h(p,q), \tag{5}$$

one gets

$$L_h = p_\phi\partial_\phi/m_0u^2 + \sqrt{-p_\phi^2 + 2m_0u(a + uE)}\,\partial_u. \tag{6}$$

To solve (3) we let $f(p_r, p_\phi, r, \phi) = \exp(F)$, where $F \equiv \lambda t - \mu\phi + w(u, E) = \text{const}$ (having taken into regard that ϕ is cyclic and that any function of p_ϕ is a constant of motion). That is, $0 = \mathrm{d}F = \partial_t F\,\mathrm{d}t + \partial_\phi F\,\mathrm{d}\phi + \partial_u F\,\mathrm{d}u \equiv \lambda\,\mathrm{d}t - \mu\,\mathrm{d}\phi + \partial_u w\,\mathrm{d}u$. To calculate $\mathrm{d}u/\mathrm{d}t = -\partial_t F/\partial_u F$, we have to set $\lambda = 1$ and $\mu = 0$, and to calculate $\mathrm{d}u/\mathrm{d}\phi = -\partial_\phi F/\partial_u F$, we have to set $\lambda = 0$ and $\mu = 1$. Thus,

$$\mathrm{d}u/\mathrm{d}t = -1/\partial_u w = \sqrt{-p_\phi^2/u^2 + 2am_0/u + 2m_0E}/m_0u, \tag{7}$$

$$\mathrm{d}u/\mathrm{d}\phi = (p_\phi/m_0u^2)/\partial_u w = (u/p_\phi)\sqrt{-p_\phi^2/u^2 + 2am_0/u + 2m_0E}. \tag{8}$$

Integration yields

$$t - t_0 = \int m_0 u\, \mathrm{d}u \Big/ \sqrt{-p_\phi^2/u^2 + 2am_0/u + 2m_0 E}, \tag{9}$$

$$\phi - \phi_0 = -p_\phi \int \mathrm{d}u \Big/ \Big(u\sqrt{-p_\phi^2/u^2 + 2am_0/u + 2m_0 E}\Big). \tag{10}$$

This is exactly what follows from the Hamilton–Jacobi equation (cf [6])

$$S \equiv -Et + p_\phi \phi + \int \sqrt{2m_0(E - a/u) - p_\phi^2/u^2}\, \mathrm{d}u = \mathrm{const}. \tag{11}$$

by differentiation w.r.t. E and p_ϕ respectively. We shall return to the just considered Hamiltonian as well as to its relativistic pendants in sections 4, 5 and 6. In the following section we shall first investigate the case of a one-dimensional quartic potential in connection with equation (2).

Remark 1. A different approach to a quantum Hamilton–Jacobi scheme with regard to the Schrödinger equation has been proposed in [7].

2. One-dimensional quartic potential

For the Hamilton function $h4 \equiv h(p, q) = p/2 + q^4/4$, the δ_h-operator reads

$$\delta_h = p\partial_q - q^3\partial_p + \hbar^2 q \partial_p^3/4 \equiv L_h + \hbar^2 q\partial_p^3/4. \tag{12}$$

By substituting $q = u$, $p = \sqrt{2E - u^4/2}$, $E \equiv h4$, we get with $f(p, q) = \exp(F)$, $F \equiv t + w(u, E) = \mathrm{const}$, and from (2)

$$\partial_u w = -1/\sqrt{2E - u^4/2} + \hbar^2 G,$$

$$G \equiv u[6(\partial_E w)^2 + (4E - u^4)(\partial_E w)^3 + 6\partial_E^2 w + 3(4E - u^4)\partial_E w \partial_E w^2 + (4E - u^4)\partial_E w^3]/8. \tag{13}$$

It follows then from $\mathrm{d}u/\mathrm{d}t = -1/\partial_u w$ that

$$t = t_0 - \int_{q_0}^{q} \partial_u w\, \mathrm{d}u = t_0 - w = t_0 + \int_{q_0}^{q} \mathrm{d}u(1/\sqrt{2E - u^4/2} - \hbar^2 G). \tag{14}$$

There seems to be no way by which a rigorous solution of the differential equation (13) could be found. (Setting $w(u, E) = \log(f(u, E))$ one gets a linear partial differential equation

$$\partial_u f = -f/\sqrt{2E - u^4/2} - (\hbar^2/8)u\big[6\partial_E^2 f + (4E - u^4)\partial_E^3 f\big],$$

which, too, offers no (at least easily to be seen) rigorous solution.) To solve (13), we shall therefore use the following successive approximation:

$$\begin{aligned} \partial_u w^{(0)} &= -1/\sqrt{2E - u^4/2}, \\ \partial_u w^{(n)} &= -1/\sqrt{2E - u^4/2} - \hbar^2 G^{(n-1)}, \qquad n \geqslant 1, \end{aligned} \tag{15}$$

where $G^{(n-1)} \equiv G(u, E, w^{(n-1)})$ is determined by equation (13). The time-dependent trajectories $u(t) \equiv q(t)$ are then obtained by integration of $\mathrm{d}u^{(n)}/\mathrm{d}t = -1/\partial_u w^{(n-1)}$. Writing for the moment $u \equiv u^{(0)}$, we have

$$\begin{aligned} G^{(0)}(u, E) = u/(512E^2\sqrt{4E - u^4})\{&\sqrt{2}u[(-2u^2 + 45u^4 - 132E)\sqrt{E} \\ &+ 6(5u^4 - 16E)E^{1/4}F(u/\sqrt{2}E^{1/4}) + 3(u^4 - 4E)F(u/\sqrt{2}E^{1/4})^2] + \sqrt{4 - u^4/E} \\ &\times E^{1/4}[-30u^2E^{3/4} - 3\sqrt{E}(2u^2 - 15u^4 + 20E)F(u/\sqrt{2}E^{1/4}) \\ &+ 3(5u^4 - 12E)E^{1/4}F(u/\sqrt{2}E^{1/4})^2 + (u^4 - 4E)(u/\sqrt{2}E^{1/4})^3]\}, \end{aligned} \tag{16}$$

where $F(u/\sqrt{2}E^{1/4}) = \mathrm{EllipticF}[\arcsin(u/\sqrt{2}E^{1/4}), -1] = -w^{(0)}(u, E)$.

Inserting for u the corresponding classical expression $u^{(0)}(t) = (4E)^{1/4}\,\mathrm{cn}[(4E)^{1/4}t]$, cn ≡ cosinus amplitudinis with module $k = 1/\sqrt{2}$, we get as a first approximation

$$\begin{aligned}\mathrm{d}u^{(1)}(t)/\mathrm{d}t &= \mathrm{d}u^{(0)}(t)/\mathrm{d}t/[1-\hbar^2\,\mathrm{d}u^{(0)}(t)/\mathrm{d}t\,G^{(0)}(u^{(0)}(t),E)+O(\hbar^4]\\ &= \mathrm{d}u^{(0)}(t)/\mathrm{d}t[1+\hbar^2\,\mathrm{d}u^{(0)}(t)/\mathrm{d}t\,G^{(0)}(u^{(0)}(t,E))]+O(\hbar^4),\end{aligned}$$

where $\mathrm{d}u^{(0)}(t)/\mathrm{d}t = \pm\sqrt{2E - u^{(0)}(t)^4/2}$.

Having no strict *a priori* estimate for the quality of this approximation one has, according to the above relation for each numerical calculation, to check whether $\hbar^2 G\,\mathrm{d}u^{(0)}(t)/\mathrm{d}t < 1$. (This criterion, which depends on the values for E, has to be applied mutatis mutandis to all examples to follow.) By letting (see section 6) $E \equiv E(n) \approx [3\Gamma(3/4)/(\sqrt{\pi}8\Gamma(5/4))2\hbar\pi(n+1/2)]^{3/4}$ it follows from numerical calculations that roughly $n \geqslant 10^5$, that is, values $E \geqslant 1.152\times10^{-25}$, satisfy this criterion. This agrees with what could have been expected, meaning that for lower energies one needs approximations of higher order. Another way to check the approximation is to calculate the difference $h(p^{(n)}(t), q^{(n)}(t)) - h(p,q)$. Numerical calculations show that despite higher oscillations of the approximate energy values $h(p^{(n)}(t), q^{(n)}(t))$ over the interval of periodicity, the mean values remain very close to the energy values for $\hbar = 0$, and that with increasing energy values classical and (approximate) Moyal trajectories differ increasingly less.

Remark 2. Using the above-defined approximation means that all higher approximations $u^{(n)}(t)$ have the same periodicity as $u^{(0)}(t)$. The question is whether this would also be true for an exact solution. We can only answer this question with regard to the results in a previous paper [4] where numerical solutions for the quartic potential had been calculated by piecewise analytic continuation. The values obtained in this way had indeed shown that the periodicity for $\hbar \neq 0$ and $\hbar = 0$ is the same. A further argument which supports the assumption that the periodicity remains unchanged is that necessarily $|q| \leqslant \sqrt{2}E^{1/4}$ for any $\hbar$ as can be seen from equation (13). Figures 1a, b and c show differences of coordinates between classical and Moyal trajectories for a quartic potential.

3. Expansion of δ_h in terms of $\hbar^{2n}$, $n \geqslant 0$

Let us now turn to the problem of calculating $\delta_h f$, $h(p,q) = T(p) + V(q)$, where (p,q) is any $2N$-tupel of canonical phase space coordinates and $V(q)$ is not necessarily a polynomial. Replacing $f(p, q)$ by its Fourier transform it suffices to calculate the action of δ_h on the corresponding Fourier character. That is,

$$\begin{aligned}&\exp[-\mathrm{i}(px+qy)]\delta_h\exp[\mathrm{i}(px+qy)]\\ &\qquad = h(p+\mathrm{i}\hbar y/2, q-\mathrm{i}\hbar x/2) - h(p-\mathrm{i}\hbar y/2, q-+\hbar x/2).\end{aligned} \tag{17}$$

Expanding this expression w.r.t. $\hbar$ and replacing the x_j and y_j by $-i\partial_{p_j}$ and $-i\partial_{q_j}$ respectively (taking if necessary into consideration a symmetrization procedure [9]) one obtains a series in terms of $\hbar^{2n}$, $n \geqslant 0$, which breaks off if $h(p, q)$ is a polynomial. A short calculation yields the following expression ($\delta(a,b) = 1$ if $a = b$ and $= 0$ if $a \neq b$ is the Kronecker symbol):

$$\begin{aligned}\delta_h f = L_h f + \sum_{n\geqslant1}\hbar^{2n}\sum_{j_1,k_1,\ldots,j_N,k_N\geqslant0}^{2n+1}&\delta(j_1+k_1+\cdots+j_N+k_N, 2n+1)\\ &\times \mathrm{i}^{j_1+k_1+\cdots+j_N+k_N-1}(-1)^{k_1+\cdots+k_N}\left(\partial_{p_1}^{j_1}\partial_{q_1}^{k_1}\ldots\partial_{p_N}^{j_N}\partial_{q_N}^{k_N}h\right)\\ &\times\left(\partial_{p_1}^{k_1}\partial_{q_1}^{j_1}\ldots\partial_{p_N}^{k_N}\partial_{q_N}^{j_N}f\right)\big/(4^n j_1!k_1!\cdots j_N!k_N!).\end{aligned} \tag{18}$$

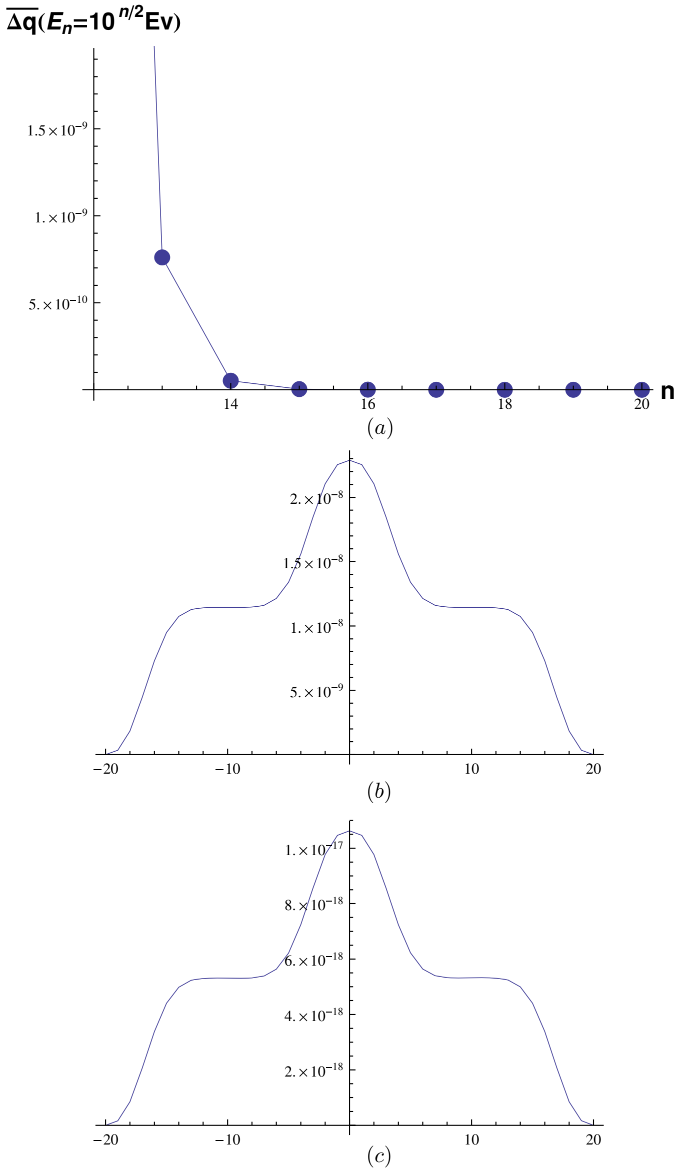

Figure 1. One-dimensional quartic potential. (*a*) Mean difference of $q(\hbar = 0) - q(\hbar > 0)$ over interval of periodicity. (*b*) $q(\hbar = 0, t) - q(\hbar \neq 0, t)$, $E = 10^6 Ev$, $q(\hbar = 0, 0) = 0.0503$. (*c*) $q(\hbar = 0, t) - q(\hbar \neq 0, t)$, $E = 10^{10}$ eV, $q(\hbar = 0, 0) = 0.503$.

Obviously δ_h is a generalization of the Liouville operator L_h the latter given by

$$L_h f = \sum_{j_1,k_1,\ldots,j_N,k_N \geqslant 0}^{1} \delta(j_1 + k_1 + \cdots + j_N + k_N, 1) \mathrm{i}^{j_1+k_1+\cdots+j_N+k_N-1} (-1)^{k_1+\cdots+k_N} \\ \times \left(\partial_{p_1}^{j_1} \partial_{q_1}^{k_1} \ldots \partial_{p_N}^{j_N} \partial_{q_N}^{k_N} h\right)\left(\partial_{p_1}^{k_1} \partial_{q_1}^{j_1} \ldots \partial_{p_N}^{k_N} \partial_{q_N}^{j_N} f\right) \big/ (4 j_1! k_1! \cdots j_N! k_N!). \tag{19}$$

Approximations of δ_h up to powers $\hbar^{2n}$ shall be denoted by $\delta_h^{(n)}$ unless $h(p, q)$ is polynomial in p and q in which case we write δ_h.

As an example we choose again $h \equiv \left(p_r^2 + p_\phi^2/r^2\right)/2m_0 + a/r$. With relations (18) and (19) one gets up to terms with $\hbar^{2n}$, $n \geqslant 2$,

$$\delta_h^{(1)} f(p_r, p_\phi, r, \phi) = \left\{p_\phi \partial_\phi / m_0 r^2 + p_r \partial_r / m_0 + \left(p_\phi^2 + a m_0 r\right)\partial_{p_r} / m_0 r^3 \right.$$
$$\left. + (\hbar^2/4m_0 r^4)\left[-r\partial_{p_r} - 3p_\phi \partial_{p_r}^2 \partial_\phi + \left(-4p_\phi^2/m_0 r + a\right)\partial_{p_r}^3\right]\right\} f(p_r, p_\phi, r, \phi).$$

Note that $\delta_h^{(1)} f(p_\phi) = 0$. It is easily proven that generally $\delta_h^{(n)} f(p_\phi) = 0$ for all $n \geqslant 0$ as could have been expected (this can generally be shown to be true for all radially symmetric Hamilton functions considered in the following).

We shall come back to this example and its relativistic pendants in the following sections. In all examples with radially symmetric Hamilton functions considered below we shall restrict ourselves to approximations of first order.

Remark 3. Let $(p_1, \ldots, p_N, q_1, \ldots, q_N) \in \mathbb{R}^{2N}$ be a set of Cartesian phase space coordinates for which $(j, k \in \{1, \ldots, N\})$;

(i) $\{p_j, q_k\}_M = -\mathrm{i}\hbar\delta_{jk}$, $\quad \{p_j, p_k\}_M = \{q_j, q_k\}_M = 0$, $\quad \{a, b\}_M \equiv \{a *_\hbar b - b *_\hbar a\}$.

Let further $(\pi_1, \ldots, \pi_N, \sigma_1, \ldots, \sigma_N) \equiv (p_r, p_{\phi_1}, \ldots, p_{\phi_{N-1}}, r, \phi_1, \ldots, \phi_{N-1})$ be a corresponding set of spherical phase space coordinates. Using the Weyl relations together with Fourier transforms it can be shown (cf [4, 5] and appendix) that relations (i) imply

(ii) $\{\pi_j, \sigma_k\}_M = -\mathrm{i}\hbar\delta_{jk}$, $\quad \{\pi_j, \pi_k\}_M = \{\sigma_j, \sigma_k\}_M = 0$.

4. Nonrelativistic and relativistic radially symmetric Coulomb potential

For the Hamilton function $h \equiv \left(p_r^2 + p_\phi^2/r^2\right)/2m_0 + a/r$ we get with $f(p_r, p_\phi, r, \phi) = \exp(F)$, $F \equiv \lambda t - \mu\phi + w(u, E) = \text{const}$, from relation (2) $\left(R \equiv -p_\phi^2 + 2m_0 u(-a + uE)\right)$

$$\partial_u w = -(\lambda m_0 u + \mu p_\phi/u)/\sqrt{R} + \hbar^2 G + O(\hbar^4), \tag{20}$$

$$G \equiv 3\mu p_\phi \partial_E w/4m_0 u^3\sqrt{R}) + \left\{3m_0 u^2\left(2p_\phi^2 + a m_0 u + p_\phi\sqrt{R}\,\mu\right)\left(\partial_E w^2 + \partial_E^2 w\right)\right.$$
$$\left. - \left(2p_\phi^2 + a m_0 u\right)\left[p_\phi^2 + 2m_0 u(a - uE)\right]\left(\partial_E w^3 + \partial_E w \partial_E^2 w + \partial_E^3 w\right)\right\}/4m_0^3 u^7. \tag{21}$$

Following the scheme outlined above for classical evolutions one has with $\lambda = 1$, $\mu = 0$,

$$\mathrm{d}t/\mathrm{d}u = -F_t/F_u = -m_0 u/\sqrt{R} - \hbar^2 G + O(\hbar^4), \tag{22}$$

and with $\lambda = 0$, $\mu = 1$,

$$\mathrm{d}\phi/\mathrm{d}u = -F_\phi/F_u = -p_\phi/u\sqrt{R} + \hbar^2 G + O(\hbar^4). \tag{23}$$

This yields with equations (20) and (21)

$$t = \int (m_0 u/\sqrt{R} - \hbar^2 G)\,\mathrm{d}u + O(\hbar^4), \tag{24}$$

$$\phi = \int (-p_\phi/u\sqrt{R} + \hbar^2 G)\,\mathrm{d}u + O(\hbar^4). \tag{25}$$

Remark 4. We might have also considered an arbitrary potential $V(r)$ in the above calculations by substituting $p_r \to \sqrt{-p_\phi^2/r^2 + 2m_0[E + V(r)]}$.

Straightforward calculation (writing now $u = r$) with a first approximation of $w(r, E) = w_0(r, E) = \arcsin\left[\left(am_0r + p_\phi^2\right)/a\sqrt{m_0\left(a^2m_0 + 2p_\phi^2E\right)}\right]$ leads to the following expression for G:

$$\begin{aligned}G \equiv G(a, m_0, p_\phi, E, \mu, \hbar)(r) &= \hbar^2\big\{p_\phi^3\left(p_\phi^2 + am_0r\right)\\ &\times\left[6p_\phi\left(2p_\phi^4 + a^2m_0^2r^2(1+\mu) + m_0p_\phi^2r(3a + 2rE\mu)\right)\right.\\ &-\left(-14p_\phi^6 + a^3m_0^3r^3(-1+3\mu) + m_0p_\phi^2r(-35a + 6rE(4+\mu)\right)\\ &\left.+am_0^2p_\phi^2r^2(6rE(2+\mu) + a(-16+3\mu))\right]\big\}/\sqrt{R}.\end{aligned} \tag{26}$$

Note that $G(a, m_0, p_\phi, E, \mu, \hbar)(r)$ does not depend on λ.

The function $G(a, m_0, p_\phi, E, \mu, \hbar)(r)$ can be exactly integrated w.r.t. r yielding the following expression:

$$\begin{aligned}\mathrm{Int}\, G \equiv \mathrm{Int}\, G(a, m_0, p_\phi, E, \mu, \hbar)(r) &= \int G(a, m_0, p_\phi, E, \mu, \hbar)(r)\,\mathrm{d}r\\ &= -\hbar^2\big\{12p_\phi^6\sqrt{-p_\phi^2 + 2m_0r(-a + rE)}\big[4\left(p_\phi^2 + am_0r\right)^3\\ &+m_0r^2\left(3p_\phi^2 + 4am_0r\right)\left(a^2m_0 + 2p_\phi^2E\right)\mu\big)\\ &+p_\phi\left(p_\phi^2 + 2m_0r(a - rE)\right)\left(56p_\phi^{10} + 3a^5m_0^5r^5(13 + 21\mu)\right.\\ &-4m_0p_\phi^8r(-28a + rE(1+9\mu)) - 2m_0^2p_\phi^6r^2(9a^2(-3+\mu)\\ &+2arE(-5+3\mu) + 6r^2E^2(1+9\mu)) + a^3m_0^4p_\phi^2r^4(-a(13+21\mu)\\ &+8rE(17+33\mu)) + 2am_0^3p_\phi^4r^3(a^2(5-3\mu)\\ &-16arE(1+3\mu) + 2r^2E^2(29+69\mu))\big]\\ &-3m_0^3r^6\left(a^2m_0 + 2p_\phi^2E\right)^2\sqrt{-p_\phi^2 + 2m_0r(-a + rE)}\\ &\times\left(2p_\phi^2E(1+9\mu) + a^2m_0(13+21\mu)\right)\\ &\times\ \arcsin\left[\left(p_\phi^2 + am_0r\right)/r\sqrt{m_0\left(a^2m_0 + 2p_\phi^2E\right)}\right])\big\}/\\ &\times 96m_0^3p_\phi^2r^6\left(a^2m_0 + 2p_\phi^2E\right)^3\sqrt{-p_\phi^2 + 2m_0r(-a + rE)}.\end{aligned} \tag{27}$$

A closer look at

$$\mathrm{Int}\, G \equiv \hbar^2\,\mathrm{Int}\, G_0\big/\left[96m_0^3p_\phi^2r^6\left(a^2m_0 + 2p_\phi^2E\right)^3\sqrt{-p_\phi^2 + 2m_0r(-a + rE)}\right]$$

shows that the integral diverges for $E \to -a^2m_0/2p_\phi^2$ (as well as for $r \to \left(a \pm \sqrt{a^2 + 2p_\phi^2E/m_0}\right)/2E$). Convergence of our approximation requires $\hbar^2 < \left(a^2m_0 + 2p_\phi^2E\right)^3$. This means that in particular the case $E = -a^2m_0/2p_\phi^2$ (this is classically a circle) needs a separate investigation. We can directly deal with this case obtaining a surprising result. Calculating $\exp(\phi)\delta_h f(r, E))\exp(-\phi)$ via Fourier transform yields $p_r\partial_r f(r, E) + K$ where K has the following property:

$$K_0 \equiv \lim_{p_r\to 0} K = \sum_{j\geqslant 0}\hbar^{2j}c_j(r, p_\phi, m_0, a)\partial_E^j f(r, E).$$

Now, $E \to E_c$ implies $p_r = \mathrm{i}\left(p_\phi^2 - am_0r\right)/p_\phi$ which means that we must set $p_\phi^2 = am_0r$, that is, $p_r = 0$ (otherwise p_r would be imaginary). The determining equation for $w(r, E) = -\log[f(r, e)]$ thus reads

$$\partial_r w = -m_0\exp(w)K/pr.$$

Hence,

$$\mathrm{d}r/\mathrm{d}\phi = -1/\partial_r w \to 0$$

if $E \to E_c$ and consequently $p_r \to 0$. This holds strictly for arbitrary $\hbar$. In other words $r(\phi) = r_c$ for all ϕ and arbitrary $\hbar$. We shall say that the case $E = E_c$ is *ħ-stable*. Note that in this exceptional example the classical and Moyal trajectories do indeed coincide (see figures 2c, 2d, 2e).

As to numerical examples we shall restrict ourselves to calculations of first order for scattering (deflection) angles. Figures 2a and b show differences of scattering angles and radii, respectively between classical and Moyal trajectories for a nonrelativistic Coulomb potential.

Now, given a function $X(a, m_0, c, p_\phi, E, r, \phi)$ in the relativistic case with $h \equiv a/r + c\sqrt{c^2m_0^2 + p_r^2 + p_\phi^2/r^2}$ one obtains its nonrelativistic correspondent as follows:

$$\mathcal{N}RX(a, m_0, p_\phi, E, r, \phi) := \lim_{c\to\infty} X(a, m_0, c, p_\phi, E + c^2m_0, r, \phi). \tag{28}$$

(Note that $h - c^2m_0 = a/r + \left(p_r^2 + p_\phi^2/r^2\right)/2m_0 + O(1/c)$.) As an example consider (cf [9]) the relativistic classical radius for $c^2p_\phi^2 > a^2$:

$$r_0(a, m0, c, p_\phi, E)(\phi) = p_\phi^2/\left[-aE + \sqrt{-c^2m_0^2\left(c^2p_\phi^2 - a^2\right) + p_\phi^2E^2}\cos\left(\sqrt{1 - a^2/c^2p_\phi^2}\,\phi\right)\right]. \tag{29}$$

Its nonrelativistic correspondent is

$$\begin{aligned} r_0(a, m0, p_\phi, E)(\phi) &\equiv \mathcal{N}Rr_0(a, m0, c, p_\phi, E)(\phi) \\ &= \lim_{c\to\infty} r_0(a, m_0, c, p_\phi, c^2m_0 + E)(\phi) \\ &= p_\phi^2/\left[-am_0 + \sqrt{m_0\left(a^2m_0 + 2p_\phi^2E\right)}\cos\phi\right], \end{aligned} \tag{30}$$

which is exactly what one gets in the classical nonrelativistic Kepler problem.

For the relativistic case we basically proceed as in the nonrelativistic case. That is, we set $f(p_r, p_\phi, r, \phi) = \exp(F), F = \lambda t - \mu\phi + w$. Writing for short $R(r) \equiv \sqrt{-c^2\left(c^2m_0^2 + p_\phi^2\right) + (a - rE)^2}$ one gets

$$\begin{aligned} \partial_r w = [&-(a - rE)\lambda + c^2p_\phi\mu/r \\ &+\hbar^2(a - rE)G(a, m_0, c, p_\phi, E, \mu)(r)]/cR(r) + O(\hbar^4). \end{aligned} \tag{31}$$

From $\mathrm{d}\phi/\mathrm{d}r = -F_r/F_\phi = \partial_r w, \lambda = 0, \mu = 1$, it follows

$$\begin{aligned} \phi(a, m_0, c, p_\phi, E, \hbar)(r) &= \phi_0(a, c, m_0, p_\phi, E)(r) \\ &+\hbar^2\int_{r_{\min}}^{r} K_r(a, m_0, c, p_\phi, E)(x)\,dx + O(\hbar^4), \end{aligned} \tag{32}$$

where $K_r(a, m_0, c, p_\phi, E)(r) \equiv (a - rE)G(a, m_0, c, p_\phi, E, \mu = 1)(r)/cR(r)$ and $\phi_0(a, c, m_0, p_\phi, E)(r)$ has to be calculated from $r_0(a, c, m0, p_\phi, E)(\phi)$. For example, for $c^2p_\phi^2 > a^2$ one has

$$\phi_0(a, c, m_0, p_\phi, E)(\infty) = \arccos\left[aE/c\sqrt{c^2m_0^2\left(a^2 - c^2p_\phi^2\right) + p_\phi^2E^2}\right]\Big/\sqrt{1 - a^2/c^2p_\phi^2}. \tag{33}$$

$r_{\min}$ is the classical minimal value $r_0(a, c, m0, p_\phi, E)(0)$.

From $\mathrm{d}r/\mathrm{d}\phi = -F_\phi/F_r, \lambda = 0, \mu = 1$ it follows after a short calculation

$$\begin{aligned} r(a, m_0, c.p_\phi, E)(\phi) &= r_0(a, m_0, c, p_\phi, E)(\phi) \\ &-\hbar^2\int K_\phi(a, c, m_0, p_\phi, E)(\phi)\,\mathrm{d}\phi + O(\hbar^4). \end{aligned} \tag{34}$$

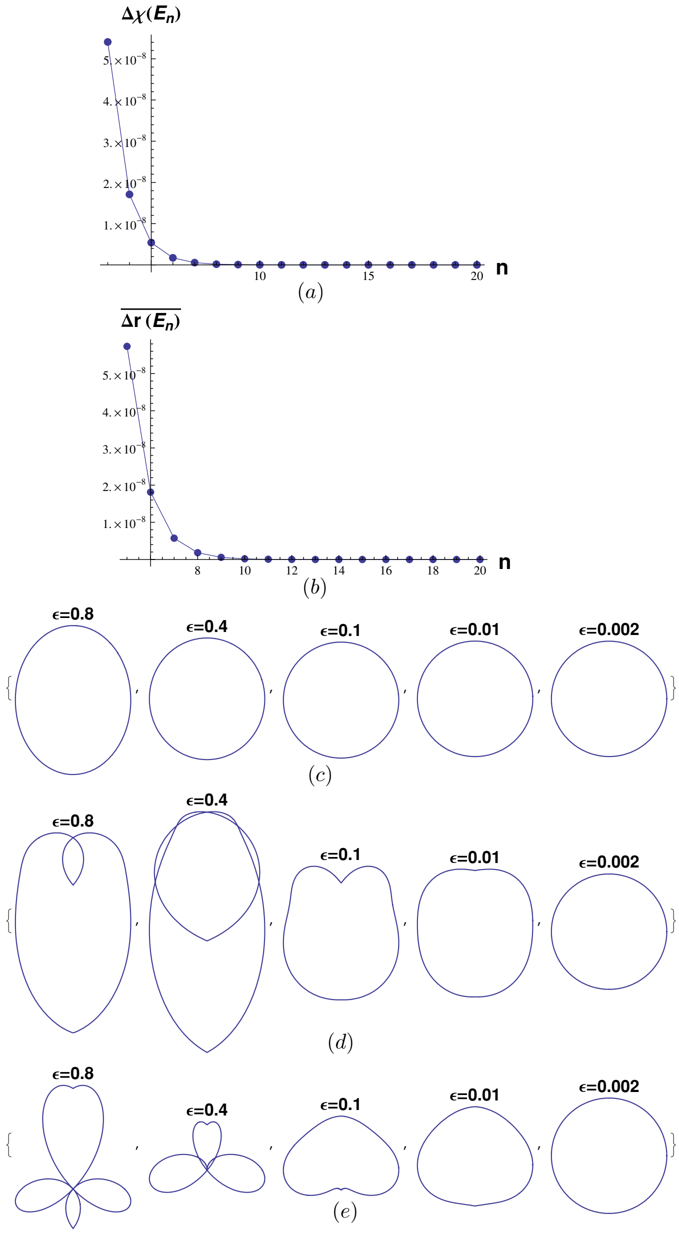

Figure 2. Nonrelativistic Coulomb potential. (*a*) Difference of deflection angles (scattering proton–proton). $\Delta\chi(E_n)/\chi(a, m_0, p_\phi, E_n, \hbar) \equiv \chi(a, m_0, p_\phi, E_n, \hbar = 0)/\chi(a, m_0, p_\phi, E_n, \hbar) - 1$. (*b*) Mean difference of radii: $\overline{\Delta r(E_n)} = \int_{\phi_{-\infty}}^{\phi_\infty} [r^{(1)}(a, m_0, p_\phi, E_n, \hbar)(\phi)/r^{(0)}(a, m_0, p_\phi, E_n, \hbar = 0)(\phi) - 1]\, \mathrm{d}\phi/(2\phi_\infty)$; $\phi_\infty \equiv \phi(r = \infty)$; $(a = a_e, m_0 = m_{\mathrm{Proton}}, p_\phi = 10^{-23})$ (cgs). (*c*, *d*, *e*) Plots for some elliptic cases with different eccentricities ϵ. (*c*) Plots of $r^{(0)}(\pm a, m_0, p_\phi, E < 0)(\phi)\,(\hbar = 0)$. (*d*) Plots of $r^{(1)}(a < 0, m_0, p_\phi, E < 0)(\phi)\,(\hbar \neq 0)$. (*e*) Plots of $r^{(1)}(a > 0, m_0, p_\phi, E < 0)(\phi)\,(\hbar \neq 0)(a = a_e, m_0 = m_{\mathrm{Proton}}, p_\phi = \sqrt{m_0(1-\epsilon^2)/(2|E|)}, E = -10^{-3}\,\mathrm{eV})$ (cgs).

In order to have escaping trajectories one must require $E^2 > c^4 m_0^2$ (otherwise $\phi_0(\infty)$ would assume complex values, except in the cases $a = m_0 = 0$). Note that the velocity for $r = \infty$ both for the nonrelativistic and relativistic case (as well as for a Schwarzschild metric) is equal to its classical correspondent $v_\infty = c\sqrt{E^2 - c^4 m_0^2}\big/E$. This can be seen from $\mathrm{d}r/\mathrm{d}t = -F_t/F_r = -\partial_r w, \lambda = 1, \mu = 0$, and the expression one gets for $G(a, m_0, c, p_\phi, E, r, \mu = 0)|_{r\to\infty}$. In contrast to the nonrelativistic case $G(a, m_0, c, p_\phi, E, r, \mu = 0)$ and $K_\phi(a, c, m_0, p_\phi, E)(\phi)$ (which can be formally integrated) are very lengthy and unwieldy, and we omit therefore to list these expressions. (It is by the way no problem also to calculate cross sections using the values obtained for $\phi(a, m_0, c, p_\phi, E)(\infty)$ and (cf [6]) the relations $\mathrm{d}\sigma/\mathrm{d}\Omega = \rho(\chi)|\mathrm{d}\rho)/\mathrm{d}\chi|/\sin(\chi)\,\mathrm{d}\Omega$, where $\mathrm{d}\Omega$ is the infinitesinal spherical angle and ρ is determined by $p_\phi = m_0 \rho v_\infty$. The expression $\mathrm{d}\rho/\mathrm{d}\chi = 1/(\mathrm{d}\chi/\mathrm{d}\rho)$ can be calculated from $\mathrm{d}\chi/\mathrm{d}\rho = \pm 2\,\mathrm{d}\phi_\infty/\mathrm{d}\rho = \pm 2(\mathrm{d}\phi_\infty/\mathrm{d}p_\phi)\,(\mathrm{d}p_\phi/\mathrm{d}\rho) = \pm 2 m_0 v_\infty\,\mathrm{d}\phi_\infty/\mathrm{d}p_\phi$.) Figures 3a, b and c illustrate some results for a relativistic Coulomb potential.

The case $m_0 = 0,\ a = 0$ delivers a somewhat strange result as shown in the following expression for a first-order approximation of the radius:

$$\begin{aligned} r(a = 0, m_0 = 0, c, p_\phi, E)(\phi) &= c p_\phi/(E\cos\phi) - E\hbar^2[(192\phi - 144\cos(2\phi) \\ &\quad - 108\cos(4\phi) - 48\cos(6\phi) - 9\cos(8\phi) + 252\sin(2\phi) + 132\sin(4\phi) \\ &\quad + 44\sin(6\phi) + 6\sin(8\phi)]\big/\big(3072 p_\phi^2\big). \end{aligned} \tag{35}$$

The nonclassical part represents a spiral from $r(\phi = 0) = 0$ to $\mathrm{i}r(\phi = \infty) = \infty$ (see section 2, figure 3(*c*)).

In the numerical examples below we have left $\hbar$ numerically unevaluated in order to show more distinctly the approximation. We have chosen values for $a = a_{\mathbf{e}} \equiv \mathbf{e}^2$ where $\mathbf{e}$ is the elementary electric charge and $a = a_{\mathbf{g}} \equiv -\kappa\, m_1 m_2$ where κ is the gravitational constant; m_1, m_2 are chosen as m_e, m_P or m_N, that is, the masses of e = electron and e^+ = positron, P = proton and N = neutron, respectively. All units are in c.g.s. or eV. Recall that $\hbar \approx 1.058 \times 10^{-26}\,\mathrm{cm}^2\ \mathrm{g}\ \mathrm{s}^{-1}$. $A \overset{\mathbf{e}}{\to} B$ or $A \overset{\mathbf{g}}{\to} B$ means particle A moves toward the target particle B. The subscripts $\mathbf{e}$ or $\mathbf{g}$ mean interaction according to $a = a_{\mathbf{e}}$ or $a = a_{\mathbf{g}}$ respectively. The deflection angle is given as $\chi = \chi_0 + \Delta\chi\,\hbar^2$ where χ_0 is the classical part. We list below only the parts $\Delta\chi_r\,\hbar^2$ and $\Delta\chi_{nr}\,\hbar^2$ where the indices r and nr stand for 'relativistic' and 'nonrelativistic' respectively. The numerical values for a, p_φ and E have been chosen such that $c^2 p_\varphi^2 > a^2$, $E^2 > c^4 m_0^2$ and that numerically $\Delta\chi\,\hbar^2 \ll 1$ which is a necessary condition for a sufficiently good approximation. The extreme smallness of the absolute numerical values of some of the parameters makes numerical calculations sometimes not sufficiently reliable. To circumvent this one can use a scaling

$$\{a, m_0, c, p_\varphi, p_r, r, r_S\} \to \{ka, km, c/\sqrt{k}, k^{3/2} p_\varphi, \sqrt{k} p_r, kr, kr_S\}$$

(r_S is the Schwarschild radius, see the following section). This scaling leaves the energy, that is, the radially symmetric Hamiltonians unchanged and multiplies the radii by the factor k. In addition the quotient $\nu = a^2\big/p_\varphi^2 c^2$ which determines for the relativistic Coulomb potential the type of trajectory—$\nu < 1$ or > 1 or $= 1$ yields elliptic or hyperbolic or parabolic trajectories respectively—remains type-invariant. This is also true for the first approximations of $\partial_r w_0(r, E, \ldots) \equiv \varphi_{\hbar=0}(r, E, \ldots)$ for Coulomb potentials. The terms G_{Coulomb} in a second approximation $\partial_r w_1(r, E, \ldots) = \partial_r w_0(r, E, \ldots) + \hbar^2 G_{\mathrm{Coulomb}} + O(\hbar^4)$ transform according to $G_{\mathrm{Coulomb}} \to G_{\mathrm{Coulomb}}/k^4$ in the nonrelativistic and $G_{\mathrm{Coulomb}} \to G_{\mathrm{Coulomb}}/k^{9/2}$ in the relativistic case, meaning that invariance of these terms requires a scaling $\hbar \to k^2\hbar$ and $\hbar \to k^{9/4}\hbar$ respectively. The difference between both cases is due to the fact that in the

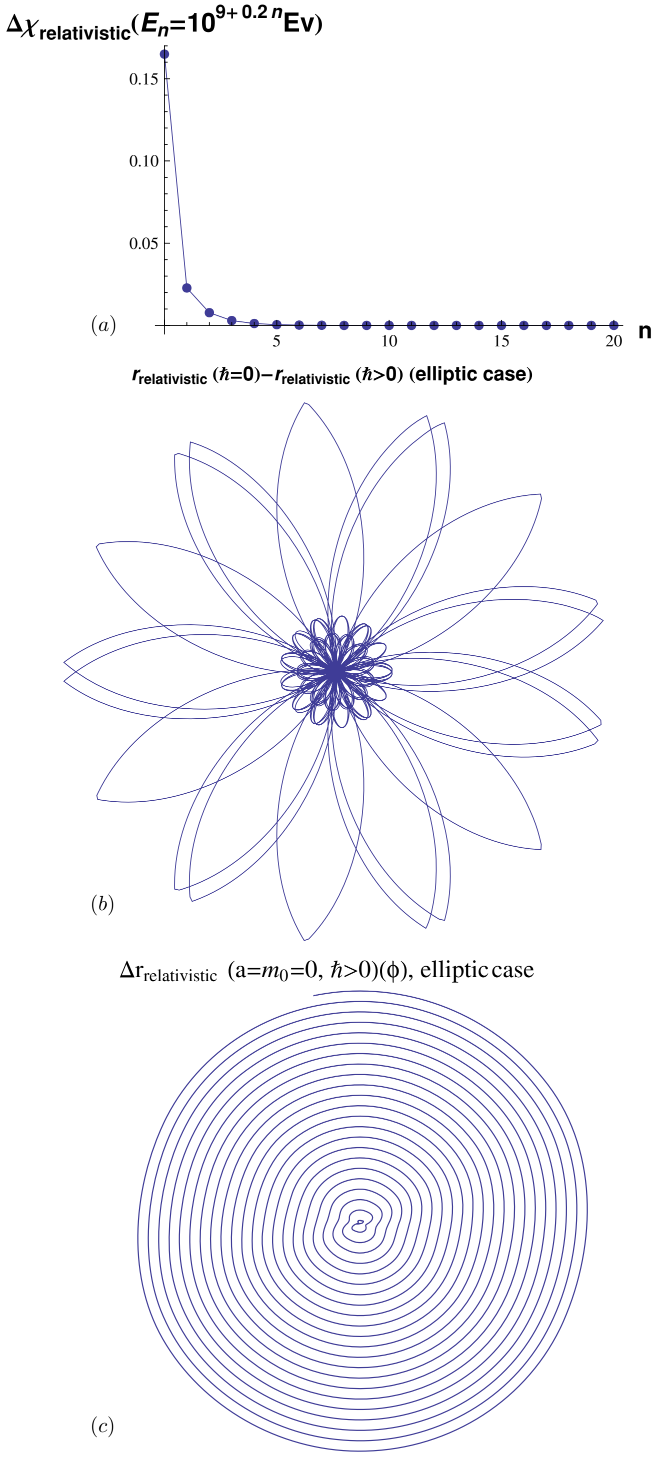

Figure 3. Relativistic Coulomb potential. (*a*) Difference of deflection angles (scattering proton–proton). (*b*) $\Delta\chi(E_n)/\chi(a, m_0, p_\phi, E_n, \hbar) \equiv \chi(a, m_0, p_\phi, E_n, \hbar = 0)/\chi(a, m_0, p_\phi, E_n, \hbar) - 1$. Trajectory of the difference of radii.

nonrelativistic case G_{Coulomb} contains only derivatives of $w_0(r, E, \ldots)$ w.r.t. E, whereas in the relativistic case derivatives of $w_0(r, E, \ldots)$ w.r.t. r and E appear. The case of a radially symmetric Schwarzschild metric which will be treated in the following section behaves exactly like the case of the radially symmetric relativistic Coulomb potential as regards the above defined scaling for $\hbar$.

In the tableau below we have listed a few values for deflection angles (see also section 9 for the figures at the end of the manuscript).

	$\Delta\chi_{nr}$	$\Delta\chi_r$	p_ϕ	E
(1) $\mathrm{e} \xrightarrow{\mathbf{e}} \mathrm{e}$:	$6.4270 \times 10^{39}\,\hbar^2$	$-9.1847 \times 10^{46}\,\hbar^2$	10^{-20}	10^8 eV
(2) $\mathrm{P} \xrightarrow{\mathbf{e}} \mathrm{P}$:	$6.4270 \times 10^{39}\,\hbar^2$	$1.8728 \times 10^{48}\,\hbar^2$	10^{-20}	10^{10} eV
(3) $\mathrm{P} \xrightarrow{\mathbf{e}} \mathrm{e}$:	$6.4270 \times 10^{39}\,\hbar^2$	$3.5525 \times 10^{47}\,\hbar^2$	10^{-20}	10^{10} eV
(4) $\mathrm{N} \xrightarrow{\mathbf{g}} \mathrm{N}$:	$1.2139 \times 10^{26}\,\hbar^2$	$1.5187 \times 10^{48}\,\hbar^2$	10^{-20}	10^{10} eV

Although the values for $\Delta\chi_{nr}$ in cases (1)–(3) are equal to each other this is not true for their classical parts; their values are $7.727\,82 \times 10^{-11}$, $3.328\,86 \times 10^{-10}$ and $-3.328\,86 \times 10^{-10}$ respectively.

We conclude with the $\hbar$-correction of the deflection angle for the space probe *Cassini* in a fly-by of the Saturn moon *Enceladus* on March 2008 with $v_\infty \approx 15$ km s^{-1} at a distance of 52 km: $|\Delta\chi| = 11.673 \times 10^{-15}\,\hbar^2$.

5. Moyal dynamics in a Schwarzschild metric

The Hamiltonian for such a metric (with motions in a plane) reads

$$h_S = c\sqrt{(1 - r_S/r)\big(c^2m^2 + p_\phi^2\big/r^2 + p_r^2(1 - r_S/r)\big)}, \tag{36}$$

where $r_S = 2\kappa M/c^2$ is the Schwarzschild (or gravitational) radius of the central mass M, and κ denotes the gravitational constant. Proceeding as in the previous sections one obtains

$$\mathrm{d}t/\mathrm{d}r = r^2E(-1 + \hbar^2G)/c(r - r_S)\sqrt{E^2r^4 - c^2r\big(c^2m_0^2r^2 + p_\phi^2\big)(r - r_S)} + O(\hbar^4), \tag{37}$$

$$\mathrm{d}\phi/\mathrm{d}r = [c^2p_\phi(r - r_S) - \hbar^2r^3EG]/ \\ c(r - r_S)\sqrt{E^2r^4 - c^2r\big(c^2m_0^2r^2 + p_\phi^2\big)(r - r_S)} + O(\hbar^4). \tag{38}$$

For $\hbar = 0$ these are exactly the relations one gets with the Hamilton–Jacobi scheme. For the function $w_{\lambda,\mu}(r, E, \ldots)$ we get

$$\partial w_{\lambda,\mu}(r, E, \ldots)/\partial r = \{-r^3E[\hbar^2G(r, E, \mu, \ldots) - \lambda] + \mu c^2p_\phi(r - r_S)\}/ \\ c\sqrt{r}(r - r_S)\sqrt{c^2(r_S - r)\big(p_\phi^2 + c^2m_0^2r^2\big) + r^3E^2}. \tag{39}$$

Thus,

$$\mathrm{d}r/\mathrm{d}\phi = [1 + \hbar^2r^2EG/c^2p_\phi(r - r_S)]\sqrt{E^2r^4 - c^2r\big(c^2m_0^2r^2 + p_\phi^2\big)(r - r_S)}/cp_\phi + O(\hbar^4), \tag{40}$$

$$\mathrm{d}r/\mathrm{d}t = c(r_S - r)(1 + \hbar^2G)\sqrt{E^2r^4 - c^2r\big(c^2m_0^2r^2 + p_\phi^2\big)(r - r_S)}/r^2E + O(\hbar^4). \tag{41}$$

We shall consider here only the case $m_0 = 0$. That is,

$\partial w_{\lambda,\mu}(r, E, \ldots)/\partial_r = \{-r^3 E[\hbar^2 G(r, E, \mu, \ldots) - \lambda] + \mu c^2 p_\phi (r - r_S)\}/c\sqrt{r}(r - r_S)\sqrt{R},$

where $R \equiv c^2(r_S - r)p_\phi^2 + r^3E^2$. There are two, essentially different, cases:

(a) $81r_S^2 > 12\omega^2, \omega \equiv p_\phi c/E$,
(b) $81r_S^2 = 12\omega^2$.

In the first case, $R = r(r - \gamma)[(r - \alpha)^2 + \beta)$, where

$$\alpha = -[34^{1/3}\omega^2 + (36\omega^4\sigma^2)^{1/3}]/4(81\omega^2\sigma)^{1/3}, \qquad \sigma = -9r_S + \sqrt{-12 + 81(r_S/\omega)^2},$$
$$\beta = -[6\omega^2 + (18\omega^4\sigma^2)^{1/3}]/[(2\sqrt{3})^5\omega^2\sigma]^{1/3}, \qquad \gamma = [23^{1/3}\omega^2 + (2\omega^4\sigma]^{1/3}/(36\omega^2\sigma)^{1/3}.$$

Denoting by $\mathcal{F}(\psi, k) = \int_0^\psi \mathrm{d}x/\sqrt{1 - k\sin^2(x)}$ the elliptic integral of first kind and by 'am' the elliptic amplitude function (that is, $u = \mathcal{F}(\psi, k) \to \psi = \mathrm{am}(u, k)$) one obtains for $E^2 > 12p_\phi^2c^2\big/81r_s^2$

$$\phi_0(r) \equiv \phi_{\hbar=0}(r) = p_\phi c\mathcal{F}[2\arctan\sqrt{B(r - \alpha)/Ar}, k], \tag{42}$$

$$r_0(\phi) \equiv r_{\hbar=0}(\phi) = -\gamma B/\{-B + A\tanh[\mathrm{am}(p_\phi c\sqrt{AB}\phi, k)]\}, \tag{43}$$

where

$$A = \sqrt{(\gamma - \alpha)^2 + \beta^2}/\gamma, \qquad B = \sqrt{\alpha^2 + \beta^2}\gamma,$$
$$k = \sqrt{1/2 + [(\alpha(\gamma - \alpha) + \beta^2]/2\gamma^2}.$$

It is easily seen that $r \to \infty$ if $\phi \to \mathcal{F}[\mathrm{arctanh}(B/A), k]/p_\phi c\sqrt{AB}$, meaning that for $E^2 > 12p_\phi^2c^2\big/81r_s^2$ we have escaping trajectories. That is, a trajectory in this case begins at infinity moving toward the black hole (circle with radius r_S), then either moves inside or stays outside (depending on the energy E) of the black hole and escapes thereafter to infinity. To give an example, let r_S be the Schwarzschild radius of a neutron and $p_\phi = 10^{-60}$ (cgs). Then the trajectory circulates twice inside the black hole for $E = 5 \times 10^{14}$ eV and circulates twice outside the black hole for $E = 5 \times 10^{13}$ eV. For the Schwarzschild radius of a neutron star the trajectory is a hyperbola which either intersects the black hole or stays away from it.

For $E^2 = 12p_\phi^2c^2\big/81r_s^2$, one has $R = (r(r + 3r_S)(r - 3r_S/2)^2$, which yields

$$\phi_0(r) = (2E/3^{3/2})\log\{[2\sqrt{3r(r + 3r_S)} + 3r_S - 4r]/(3r_S - 2r)\}, \tag{44}$$

$$r_0(\phi) = r_0(-\phi) = (3r_S/2)[1 + \exp(3^{3/2}\phi/2E)]^2/[1 - 4\exp(3^{3/2}\phi/2E) + \exp(3^{3/2}\phi/E)]. \tag{45}$$

The allowed intervals for ϕ are $D_+ = (\phi_\infty, \infty]$ and $D_- = [-\infty, -\phi_\infty)$, where $\phi_\infty = (2E/3^{3/2})\log(2+\sqrt{3})$. That is, $\pm\phi_\infty$ are the angles with which the trajectories arrive at infinity. Because of $r_0(\phi) = r_0(-\phi)$, it does not matter which interval we consider. Take for example D_+. Then the trajectory of an object with rest mass $m = 0$ starts at infinity ($\phi = \phi_\infty$ is a pole of first order), moves toward a circle with radius $3r_S/2$ which it approaches asymptotically ($\phi \to \infty$). The following list of values for $r_0(\phi)$, $E = 10^{13}$ eV, for $80 \leqslant \phi \leqslant 100$ in steps of 1 provides an example:

$$(1.500\,02r_S, 1.500\,02r_S, 1.500\,01r_S, 1.500\,01r_S, 1.500\,01r_S,$$
$$1.500\,01r_S, 1.500\,01r_S, 1.500\,01r_S, 1.500\,01r_S, 1.5r_S, 1.5r_S,$$
$$1.5r_S, 1.5r_S, 1.5r_S, 1.5r_S, 1.5r_S, 1.5r_S, 1.5r_S, 1.5r_S, 1.5r_S, 1.5r_S).$$

Using the foregoing relations it follows with a first approximation for $G(r, E, \ldots)$ that $\lim_{r\to\infty} \mathrm{d}r/\mathrm{d}t = -c\sqrt{E^2 - c^4 m_0^2}/E$. That is, we obtain the same expression as for a relativistic Coulomb potential.

A first approximation for $\phi(r)$ is

$$\phi(r) = \phi_{\hbar=0}(r) - (\hbar^2 E/c) \int r^3 K(r, E, \ldots)/ \\ (r - r_S)\sqrt{c^2(r_S - r)\left(p_\phi^2 + c^2 m_0^2 r^2\right) + r^3 E^2}\, \mathrm{d}\, r, \tag{46}$$

where $K(r, E, \ldots) = G(r, E, w_{\hbar=0}(r, E, \ldots), \ldots) = G(r, E, \phi_{\hbar=0}(r), \ldots)$. For $r(\phi)$ a first approximation reads

$$r(\phi) = r_0(\phi) + \left(\hbar^2 E / c^3 p_\phi^2\right) \int r^3 K(r, E, \ldots)\, \mathrm{d}\phi_0/\mathrm{d}r \\ \times \sqrt{c^2(r_S - r)\left(p_\phi^2 + c^2 m_0^2 r^2\right) + r^3 E^2}/(r - r_S)\, \mathrm{d}r. \tag{47}$$

The time dependence is given by

$$r(t) = r_0(t) + (\hbar^2 c/E) \int \mathrm{d}t/\mathrm{d}r_0 (r - r_S) K(r, E, \ldots) \\ \times \sqrt{c^2(r_S - r)\left(p_\phi^2 + c^2 m_0^2 r^2\right) + r^3 E^2}/r^2\, \mathrm{d}r. \tag{48}$$

As in the case of a Coulomb potential the integrals have to be numerically evaluated. A more detailed study of Moyal trajectories in a Schwarzschild metric shall be given in a paper to follow.

6. Eigenvalues: a generalization of the Bohr–Sommerfeld formula

For a classical Hamilton system $h(p, q), (p, q) \in \mathbb{R}^2$, with periodic motion the Bohr–Sommerfeld formula (cf [10])

$$\oint p\, \mathrm{d}q = 2\pi\hbar(n + 1/2) \tag{49}$$

provides a quasiclassical approximation of eigenvalues E_n for $n \gg 1$. For example for $h(p, q) = p^2/2 + q^4/4$ this formula delivers

$$\oint p\, \mathrm{d}q = \int_{-\sqrt{2}E^{1/4}}^{\sqrt{2}E^{1/4}} \sqrt{2E - u^4/2}\, \mathrm{d}u = \sqrt{\pi}(8/3)E^{3/4}\Gamma(5/4)/\Gamma(3/4). \tag{50}$$

That is, $E_n \sim n^{4/3}, n \gg 1$. We conjecture that (51) holds exactly if the classical expression $p \equiv \mathrm{d}q/\mathrm{d}t$ is replaced by the corresponding expression in a Moyal algebra as obtained from the extended Hamilton–Jacobi relations given above. For the quartic potential we get from relation (13)

$$p_\hbar = -1/\partial_u w = \sqrt{2E - u^4/2}/(1 - \hbar^2\sqrt{2E - u^4/2}\, G). \tag{51}$$

Using the approximation $G^{(0)}$ defined by (15) and replacing the integration by a sum with steps $j/\sqrt{2}E^{1/4}$, $-M \leqslant j \leqslant M$, we obtain

$$I_\hbar \equiv \oint p_\hbar\, \mathrm{d}q \approx b_0 E^{3/4} + -\hbar^2 E^{3/4} \sum_{j=-M}^{M} a_j/[1 - \hbar^2(b_{j1}E^{-1} + b_{j2}E^{-5/4} + b_{j3}E^{-3/2})] + O(\hbar^4) \\ \approx \alpha_0 E^{3/4} - \hbar^2(\alpha_1 E^{-1/4} + \alpha_2 E^{-1/2} + \alpha_3 E^{-3/4}) + O(\hbar^4). \tag{52}$$

For example for $M = 40$ one has

$$2\pi\hbar(n+1/2) \approx 3.489\,52E_n^{3/4} - \hbar^2\big(0.167\,07E_n^{-3/4} - 0.556\,90E_n^{-1/2} + 0.412\,34E_n^{-1/4}\big) + O(\hbar^4). \tag{53}$$

Although the foregoing calculations are rather crude approximations, numerical tests up to the order 10^{20} for n show that E_n/E_{n+1} tends to 1 (alternatively $E_{n+1} - E_n$ tends to 0) if n tends to ∞. (Note that (54) has three real solutions for E_n. The correct values are those which increase if n grows, the other ones decrease with growing n.)

Remark 5. For a radially symmetric potential with closed trajectories the Bohr–Sommerfeld formula reads $\oint p_r\,\mathrm{d}r = 2\pi\hbar(n_r + 1/2)$, where n_r is the radial quantum number (cf [9]).

Remark 6. The extension of the Bohr–Sommerfeld quantization, namely the calculation of eigenvalues for discrete simple spectra has been treated over the past 25 years by many authors in different approaches; we refer to [11] for an extensive discussion and references.

7. Eigenfunctions and eigenvalues of δ_h

It is well known that for the linear oscillator $h(p,q) = (p^2+q^2)/2$ the Wigner transforms are

$$W_{m,n}(p,q) := \sqrt{2/\hbar\pi}\int_{\mathbb{R}} \exp(2\mathrm{i}py/\hbar)\psi_m(q+y)\psi_n(q-y)\,\mathrm{d}y,$$

where the $\psi_n, n \in \mathbb{N}_0$, are orthonormal eigenfunctions of the Hamilton operator of the linear oscillator and form an o.n. set of eigenfunctions of δ_h with eigenvalues $\omega_{m,n} = \hbar(m-n)$ (these are actually frequencies). We are going to show that generally the functions

$$W_{f,g}(p,q) := (2/\hbar\pi)^{n/2}\int_{\mathbb{R}^n} \exp(2\mathrm{i}py/\hbar) f(q+y)g(q-y)\,\mathrm{d}y, \qquad (p,q)\in\mathbb{R}^{2n},$$

form an o.n. set of eigenfunctions of δ_h, $h(p,q) = T(p) + V(q)$, $(p,q) \in \mathbb{R}^{2n}$, provided f and g belong to an o.n. set of eigenfunctions of $H = T(\mathrm{i}\hbar\partial_u) + V(u)$, $u = (u_1,\ldots,u_n) \in \mathbb{R}^n$. The eigenvalues of δ_h are then $\omega_{f,g} := E_f - E_g$, where $Hf = E_f f$, $\bar{H}g = E_g g$. We consider first the one-dimensional case. Let $T(p) = \int_{\mathbb{R}} FT(z)\exp(-\mathrm{i}zp)\,\mathrm{d}z$ and

$$K_{f,g}(p,q,y,\eta) := \exp(2py/\hbar) f(q+\eta)g(q-\eta),$$

and let Ff and Fg denote the Fourier transforms of f and g respectively. Then a short calculation yields

$$\begin{aligned}
&\int_{\mathbb{R}}\left(\int_{\mathbb{R}} \delta_{T(p)}K_{f,g}(p,q,y-x,y)\,\mathrm{d}p\right)\mathrm{d}y \\
&\quad= \int_{\mathbb{R}} FT(z)\left[\int_{\mathbb{R}}\left(\int_{\mathbb{R}} \delta_{\exp(-\mathrm{i}pz)}K_{f,g}(p,q,y-x,y)\,\mathrm{d}p\right]\mathrm{d}z \\
&\quad= \pi\hbar\bigg[g(q-x)\int_{\mathbb{R}} T(\hbar u)\exp[-\mathrm{i}(q+x)u]Ff(u)\,\mathrm{d}u \\
&\qquad + f(q+x)\int_{\mathbb{R}} T(-\hbar v)\exp[-\mathrm{i}(q-x)v]Fg(v)\,\mathrm{d}v\bigg] \\
&\quad= \pi\hbar[-g(q-x)T(\mathrm{i}\hbar\partial_{q+x})f(q+x) + f(q+x)T(-\mathrm{i}\hbar\partial_{q-x})g(q-x)].
\end{aligned}$$

We have further

$$\int_{\mathbb{R}}\left(\int \delta_{V(q)}K_{f,g}(p,q,y-x,y)\,\mathrm{d}p\right)\mathrm{d}y = \pi\hbar f(q+x)g(q-x)[V(q-x) - V(q+x)]$$

and

$$\int_{\mathbb{R}} \left(\int K_{f,g}(p,q,y-x,y)\,\mathrm{d}p \right) \mathrm{d}y = \pi\hbar f(q+x)g(q-x).$$

Thus,

$$\begin{aligned}\int_{\mathbb{R}} \left(\int (\delta_{T(p)+V(q)} - \omega_{f,g}) K_{f,g}(p,q,y-x,y)\,\mathrm{d}p \right) \mathrm{d}y/\pi\hbar f(q+x) \\ = -[T(\mathrm{i}\hbar\partial_{q+x} f(q+x)]/f(q+x) - V(q+x) \\ + [T(-\mathrm{i}\hbar\partial_{q-x} g(q-x)]/g(q-x) + V(q-x).\end{aligned}$$

Hence, if $H := T(\mathrm{i}\hbar\partial_u) + V(u)$ and $\bar{H} := T(-\mathrm{i}\hbar\partial_u) + V(u)$, and if f and g satisfy $Hf = E_f f$ and $\bar{H}g = E_g g$ respectively then $(\delta_{h(p,q)} - \omega_{f,g})W_{f,g}(p,q) = 0, \omega_{f,g} = E_f - E_g$. The extension to more dimensions (using Cartesian coordinates for convenience) is obvious. As an example we choose the relativistic radially symmetric Hamilton function $h(p,q) = c\sqrt{c^2m_0^2 + \left(p_1^2 + p_2^2\right)/2m_0} + V\left(\sqrt{q_1^2 + q_2^2}\right)\left(=c\sqrt{c^2m_0^2 + \left(p_r^2 + p_\phi^2/r^2\right)/2m_0} + V(r)\right)$. The corresponding Hamilton operator is then $H = \bar{H} = c\sqrt{c^2m_0^2 - (\hbar^2/2m_0)\Delta_u} + V(|u|)$.

Remark 7. Replacing in the above calculations g by its complex conjugate $\bar{g}$ the second eigenvalue equation reads $H\bar{g} + V\bar{g} = E_{\bar{g}}\bar{g}$.

Remark 8. The role of the Wigner function in connection with the Schrödinger equation has been studied in numerous papers (see for example [12, 13] and [14]). So we do not claim to have provided in this section something genuinely new. We just wanted to reiterate the connection between the eigenvalues and eigenfunctions of the operator δ_h and the corresponding expressions of the associated Schrödinger equation.

8. Conclusion

Moyal dynamics as treated here looks at a first glance like an extension of classical dynamics determined by a parameter $\hbar$. That is, trajectories in Moyal dynamics, in short: Moyal trajectories, appear in terms of r(=radius), ϕ(=angle) and t(=time) like 'classical' objects. This is in a strict sense not correct. Although we did not work with a 'Moyal quantization', namely a Hilbert space representation of a Moyal algebra [5, 8], our setup is based on a noncommutative product, namely the the star product $*_\hbar$, which provides the basis for Moyal dynamics in phase space. There is however a connection between both aspects as demonstrated by the above-presented generalization of the Bohr–Sommerfeld formula, which by conjecture delivers the exact eigenvalues of stationary states in the Schrödinger picture. This is somewhat also expressed by the $\hbar$-stability of circular motions, meaning that for a (nonrelativistic) radially symmetric Coulomb potential the classical trajectory of a circle $\left(E = -m_0a^2/2p_\phi^2\right)$ coincides (up to the scaling factor $\hbar$) with the corresponding Moyal trajectory (this seems so far the only nontrivial example in which both trajectories coincide). It remains to be shown whether trajectories in a Hilbert space representation and in the scheme considered here are comparable. It is strange (and we agree here with the authors of [3]) that very little has been published on the subject presented in this paper, whereas there exists an abundance of literature dealing with quantum mechanics (in a strict sense) in phase space as outlined by the papers of Moyal, Groenewold, Wigner *et al* [1].

9. Figures

The underlying data of all figures have been chosen such that the above-explained (somewhat ad hoc) convergence criterion $\hbar^2 G < 1$ (with G being the relevant term in the corresponding expansion in powers of $\hbar^{2n}$, $n \geqslant 1$) is satisfied. Physical numerical values (as far as they are used) are given in the cgs sytem. All figures show what could have been expected with the exception of figures 2(*d*) and (*e*) which look rather strange for higher eccentricities ϵ. Whereas figure 2(*c*) ($\hbar = 0$) looks the same for positive and negative values of a, this is obviously not the case for $\hbar \neq 0$. However (see section 4), and this is confirmed by numerical calculations, one has in any case $\lim_{\epsilon\to 0}\left[r_\epsilon^{(1)}(\phi+\phi_0) - r_\epsilon^{(0)}(\phi)\right] = 0$ for all ϕ (where ϕ_0 is a possible constant phase depending on the sign of a). In figure 3(*b*) we had due to intrinsic numerical problems of this particular example to use a scaling in order to show the typical characteristics, so that numerical values for physical parameters have been omitted.

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Appendix

For two dimensions the connections between Cartesian and radially symmetric phase space coordinates w.r.t. a Moyal quantization can be established in a more direct way. Note first that $\{p_1, p_2, q_1, q_2\} = \{p_r\cos(\phi) - p_\phi\sin(\phi)/r, p_r\sin(\phi) + p_\phi\cos(\phi)/r, r\cos(\phi), r\sin(\phi)\}$ and $\{p_r,\ p_\phi,\ r,\ \cos(\phi), \sin(\phi)\} = \left\{(p_1q_1 + p_2q_2)\big/\sqrt{q_1^2+q_2^2},\ p_2q_1 - p_1q_2,\ \sqrt{q_1^2+q_2^2},\ q_1\big/\sqrt{q_1^2+q_2^2}, q_2\big/\sqrt{q_1^2+q_2^2}\right\}$.

Define then the operators

$$
\begin{aligned}
O_M(p_r) &= p_r - \mathrm{i}(\hbar/2)(\partial_r q_1\partial_{q_1} + \partial_r q_2\partial_{q_2}) = p_r - \mathrm{i}(\hbar/2)(\cos(\phi)\partial_{q_1} + \sin(\phi)\partial_{q_2}),\\
O_M(p_\phi) &= p_\phi - \mathrm{i}(\hbar/2)(\partial_\phi q_1\partial_{q_1} + \partial_\phi q_2\partial_{q_2}) = p_\phi - \mathrm{i}(\hbar/2)(-\sin(\phi)\partial_{q_1} + \cos(\phi)\partial_{q_2}),\\
O_M(r) &= r + \mathrm{i}(\hbar/2)(\partial_{p_r} p_1\partial_{p_1} + \partial_{p_r} p_2\partial_{p_2}) = r + \mathrm{i}(\hbar/2)(\cos(\phi)\partial_{p_1} + \sin(\phi)\partial_{p_2}),\\
O_M(\phi) &= \phi + \mathrm{i}(\hbar/2)(\partial_\phi p_1\partial_{p_1} + \partial_\phi p_2\partial_{p_2}) = r + \mathrm{i}(\hbar/2r)(-\sin(\phi)\partial_{p_1} + \cos(\phi)\partial_{p_2}).
\end{aligned}
$$

Inserting now for $\{p_r, p_\phi, r, \cos(\phi), \sin(\phi)\}$ the above expressions in terms of p_1, p_2, q_1, q_2 and taking into regard that $\{p_j, p_k\}_M = \{q_j, q_k\}_M = 0$ and $\{p_j, q_k\}_M = -\mathrm{i}\hbar\delta_{jk}$, one gets after a short calculation

$$
\begin{aligned}
&\{O_M(p_r), O_M(r)\}_M = \{O_M(p_\phi), O_M(\phi)\}_M = -\mathrm{i}\hbar\\
&\{O_M(p_r), O_M(p_\phi)\}_M = \{O_M(p_\phi), O_M(r)\}_M =\\
&\{O_M(p_r), O_M(\phi\}_M = \{O_M(r), O_M(\phi)\}_M = 0.
\end{aligned}
$$

That is, we can set (as could have been expected from the foregoing definitions)

$$
\begin{aligned}
O_M(p_r) &= p_r - \mathrm{i}\hbar\partial_r/2, \qquad & O_M(p_\phi) &= p_\phi - \mathrm{i}\hbar\partial_\phi/2,\\
O_M(r) &= r + \mathrm{i}\hbar\partial_{p_r}/2, \qquad & O_M(\phi) &= \phi + \mathrm{i}\hbar\partial_{p_\phi}/2.
\end{aligned}
$$

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